Last time:

$$\Gamma_{N,\infty}[\chi, K] = \Gamma_{N,\infty}[\chi, 0] + \int d^4x\, \mathcal{D}_N^n[\chi; x]\, K_n(x),$$

$$S_R[\chi, K] = S_R[\chi] + \int d^4x\, \Delta^n[\chi; x]\, K_n(x).$$

Then equation

$$(S_R, \Gamma_{N,\infty}) = 0$$

becomes

$$(1)\quad \int d^4x \left[\Delta^n(\chi; x) \frac{\delta_L \Gamma_{N,\infty}[\chi, 0]}{\delta \chi^n(x)} + \mathcal{D}_N^n[\chi; x] \frac{\delta_L S_R[\chi]}{\delta \chi^n(x)} \right] = 0$$

and

$$(2)\quad \int d^4x \left[\Delta^n(\chi; x) \frac{\delta_L \mathcal{D}_N^m(\chi; y)}{\delta \chi^n(x)} + \mathcal{D}_N^n(\chi; x) \frac{\delta_L \Delta^m(\chi; y)}{\delta \chi^n(x)} \right] = 0$$

Now introduce the quantities

$$\Gamma_N^{(\varepsilon)}[\chi] \equiv S_R[\chi] + \varepsilon \Gamma_{N,\infty}[\chi, 0]$$

and

$$\Delta_N^{(\varepsilon)n}(x) \equiv \Delta^n(x) + \varepsilon \mathcal{D}_N^n(x)$$

with ε infinitesimal.

Then equation (1) tells us that $\Gamma_N^{(\varepsilon)}[\chi]$ is invariant under the trf.

$$\chi^n(x) \mapsto \chi^n(x) + \theta \Delta_N^{(\varepsilon)n}(x), \qquad (3)$$

while eq. (2) gives together with $\delta_\theta \Delta^u(x)=0$ that trf. (3) is nilpotent!

What is this nilpotent trf.?

$T_{N,\infty}$ consists of terms of dim. 4 or less (power-counting renormalizable)

$\longrightarrow$ • $\mathcal{D}^u_N$ and $\Delta^{(\varepsilon)u}_N(x)$ have at most dim of original BRST trf. $\Delta^u(x)$.

• also same Lorentz trf. properties and ghost numbers as $\Delta^u(x)$.

$\longrightarrow$ most general form of trf. (3):

$$\psi \longmapsto \psi + i\theta\omega^\alpha T_\alpha \psi ,$$

$$A_{\alpha\mu} \longmapsto A_{\alpha\mu} + \theta\left[B_{\alpha\beta}\partial_\mu \omega_\beta + D_{\alpha\beta\gamma} A_{\beta\mu}\omega_\gamma\right],$$

$$\omega_\alpha \longmapsto \omega_\alpha - \frac{1}{2}\theta E_{\alpha\beta\gamma}\omega_\beta\omega_\gamma ,$$

where T_α is some matrix, and $B_{\alpha\beta}$, $D_{\alpha\beta\gamma}$, and $E_{\alpha\beta\gamma}$ are constants ($E_{\alpha\beta\gamma}$ anti-sym. in β,γ).

Further, $\omega^*_\alpha \longmapsto \omega^*_\alpha - \theta h_\alpha$, $h_\alpha \longmapsto h_\alpha$ (linear, hence unchanged).

Imposing condition of nilpotence gives

- $E_{\alpha\beta\gamma} E_{\beta\delta\varepsilon} + E_{\alpha\beta\varepsilon} E_{\beta\gamma\delta} + E_{\alpha\beta\delta} E_{\beta\varepsilon\gamma} = 0$

  $\longrightarrow \quad E_{\alpha\beta\gamma} = a\, C_{\alpha\beta\gamma}$, a constant

- $D_{\alpha\beta\gamma} D_{\beta\delta\varepsilon} - D_{\alpha\beta\varepsilon} D_{\beta\delta\gamma} = a\, C_{\beta\varepsilon\gamma} D_{\alpha\delta\beta}$

  $\longrightarrow \quad D_{\alpha\beta\gamma} = b\, C_{\alpha\beta\gamma}$

- $B_{\alpha\beta} E_{\beta\gamma\delta} = D_{\alpha\beta\delta} B_{\beta\gamma} \longrightarrow B_{\alpha\beta} = ab\, \delta_{\alpha\beta}$

  (b const.)

- $[T_\beta, T_\gamma] = i E_{\alpha\beta\gamma} T_\alpha \longrightarrow T_\alpha = a t_\alpha$

Summarizing, we get the following trfs.

$$
\begin{aligned}
& \psi \longmapsto \psi + i a \theta \omega^\alpha t_\alpha \psi , \\
& A_{\alpha\mu} \longmapsto A_{\alpha\mu} + a\theta \left[b\, \partial_\mu \omega_\alpha + C_{\alpha\beta\gamma} A_{\beta\mu} \omega_\gamma \right], \\
(4) \quad & \omega_\alpha \longmapsto \omega_\alpha - \frac{1}{2} a \theta\, C_{\alpha\beta\gamma} \omega_\beta \omega_\gamma , \\
& \omega^*_\alpha \longmapsto \omega^*_\alpha - \theta h_\alpha , \\
& h_\alpha \longmapsto h_\alpha .
\end{aligned}
$$

Next, we want to use this symmetry to constrain

$$
T^{(\varepsilon)}_N = \int d^4x\, \mathcal{L}^{(\varepsilon)}_N
$$

with $\mathcal{L}^{(\varepsilon)}_N$ a local function of dim ≤ 4.

$\mathcal{L}_N^{(\varepsilon)}$ must be invariant under all "linear" symmetries of original Lagrangian:

$$\mathcal{L}_{NEW} = \mathcal{L}_M - \frac{1}{4} F_\alpha^{\mu\nu} F_{\alpha\mu\nu} - \partial_\mu \omega_\alpha^* \partial^\mu \omega_\alpha$$
$$+ C_{\alpha\beta\gamma} (\partial_\mu \omega_\alpha^*) A_\gamma^\mu \omega_\beta + h_\alpha \partial_\mu A_\alpha^\mu + \frac{1}{2} \xi h_\alpha h_\alpha .$$

(i) Lorentz invariance

(ii) Global gauge invariance

$$\delta \psi_\ell (x) = i\varepsilon^\alpha (t_\alpha)_\ell{}^m \psi_m (x),$$
$$\delta A^\beta{}_\mu (x) = C_{\beta\gamma\alpha} \varepsilon^\alpha A^\gamma{}_\mu (x),$$
$$\delta \omega_\beta (x) = C_{\beta\gamma\alpha} \varepsilon^\alpha \omega_\gamma (x),$$
$$\delta \omega_\beta^* (x) = C_{\beta\gamma\alpha} \varepsilon^\alpha \omega_\gamma^* (x),$$
$$\delta h_\beta (x) = C_{\beta\gamma\alpha} \varepsilon^\alpha h_\gamma (x),$$

with const. infinitesimal ε^α.

(iii) Antighost translation invariance

$$\omega_\alpha^* (x) \longrightarrow \omega_\alpha^* (x) + \underbrace{c_\alpha}_{\text{const.}} ,$$

(iv) Ghost number conservation
(+1 for ω_α, −1 for ω_α^*, 0 for all other fields)

$\rightarrow$ construct most general Lagrangian $\mathcal{L}_N^{(\varepsilon)}$ that is of dim. 4 or less, invariant under above linear trfs., and BRST invariant under (4):

(iii) $\rightarrow$ ω_α^* appears only in the for $\partial_\mu \omega_\alpha^*$

(iv) $\rightarrow$ ω_α and ω_α^* appear in pairs

$\omega_\alpha \partial_\mu \omega_\beta^*$

(i) $\rightarrow$ $\omega_\alpha \partial_\mu \omega_\beta^*$ is not Lorentz invariant, so we must take

$\partial_\mu \omega_\alpha^* \partial^\mu \omega_\beta$ or $\partial_\mu \omega_\alpha^* A_\gamma^\mu \omega_\beta$

(i) $\rightarrow$ h_α is of dim +2, so we get

$h_\alpha h_\alpha$ or $h_\alpha A_\beta^\mu A_{\gamma\mu}$ or $h_\alpha \partial_\mu A_\alpha^\mu$

$\mathcal{L}_N^{(\varepsilon)}$ will also contain renormalizable terms involving only matter and gauge fields.

$\rightarrow$ denote by $\mathcal{L}_{\psi, A}$

(ii) $\rightarrow$ most general form:

$$\mathcal{L}_N^{(\varepsilon)} = \mathcal{L}_{\psi A} + \frac{1}{2}\xi' h_\alpha h_\alpha + c h_\alpha \partial_\mu A_\alpha^\mu - e_{\alpha\beta\gamma} h_\alpha A_\beta^\mu A_{\gamma\mu}$$

$$\text{(5)} \qquad - Z_\omega (\partial_\mu \omega_\alpha^*)(\partial_\mu \omega_\alpha) - d_{\alpha\beta\gamma} (\partial_\mu \omega_\alpha^*) \omega_\beta A_\gamma^\mu,$$

where ξ', Z_ω, c, $d_{\alpha\beta\gamma}$, and $e_{\alpha\beta\gamma}$ are unknown constants

Imposing BRST invariance on (5) gives:

- cancellation of $\theta \partial_\mu h_\alpha \partial^\mu \omega_\alpha$

  $\longrightarrow \quad c = Z_\omega / ab$

- cancellation of $\theta \partial_\mu h_\alpha \omega_\beta A_\gamma^\mu$

  $\longrightarrow d_{\alpha\beta\gamma} = -(Z_\omega/b)\, C_{\alpha\beta\gamma}$

- $\theta \partial_\mu \omega_\alpha^* \omega_\beta \omega_\gamma A_\delta^\mu$ automatically cancel by Jacobi identity

- cancellation of $\theta h_\alpha \partial_\mu \omega_\beta A_\gamma^\mu$ gives

  $e_{\alpha\beta\gamma} = 0$.

- the effect of $\Delta^{(\varepsilon)}$ BRST-trf. on $\mathcal{L}_{\varphi,A}$ is a gauge trf. with par.

  $\varepsilon_\alpha = ab\theta\omega_\alpha$ and gauge couplings renormalized by $\frac{1}{b}$

  (i.e. $t_\alpha \mapsto \tilde{t}_\alpha = t_\alpha/b$, $C_{\alpha\beta\gamma} \mapsto \tilde{C}_{\alpha\beta\gamma} \equiv C_{\alpha\beta\gamma}/b$)

  $\longrightarrow \mathcal{L}_{\varphi,A}$ is <u>gauge invariant</u>

→ most general renormalizable Lagrangian allowed by the symmetries

$$\mathcal{L}_N^{(\varepsilon)} = -\frac{1}{4} Z_A \tilde{F}_\alpha^{\mu\nu} \tilde{F}_{\alpha\mu\nu} - Z_\psi \bar{\psi}\gamma^\mu [\partial_\mu - i t_\alpha A_{\alpha\mu}] \psi$$
$$- \bar{\psi} m_\psi \psi + \frac{1}{2} \xi' h_\alpha h_\alpha + (Z_\omega / ba) h_\alpha \partial_\mu A_\alpha^\mu$$
$$- Z_\omega (\partial_\mu \omega_\alpha^*)(\partial_\mu \omega_\alpha) + Z_\omega C_{\alpha\beta\gamma} (\partial_\mu \omega_\alpha^*) \omega_\beta A_\gamma^\mu$$

→ apart from a few new coefficients, this is the same Lagrangian as before!

→ infinities can be absorbed in counterterms of original Lagrangian